EE105
Microelectronic Devices and Circuits

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Linear Time-Invariant (LTI) System

- Response of a system

$$x(t) \longrightarrow \boxed{\text{H}} \longrightarrow y(t)$$

- The system is linear if

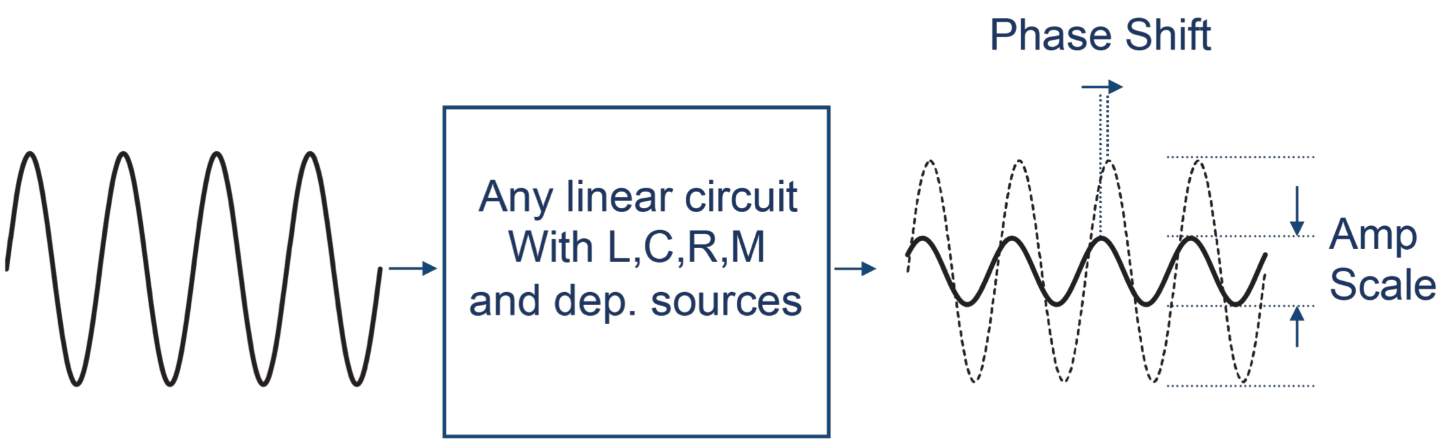
$$a_1 x_1(t) + a_2 x_2(t) \longrightarrow \boxed{\text{H}} \longrightarrow a_1 y_1(t) + a_2 y_2(t)$$

- The system is time-invariant if

$$x(t+T) \longrightarrow \boxed{\text{H}} \longrightarrow y(t+T)$$

What's Nice about LTI System?

- **Can use superposition**
- **Easy conversion between time and frequency response**
- **Most systems in real life are LTI systems**
  - **Focus of this class**

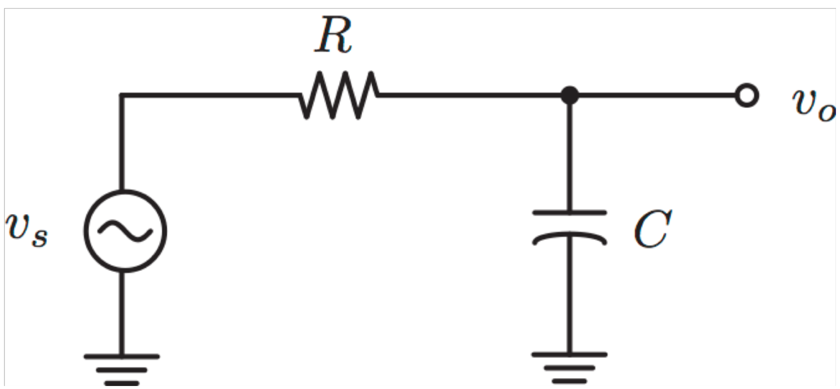


Example: Low Pass Filter (LPF)

- **Input signal:** $v_s(t) = V_s \cos(\omega t)$
- **We know that:** $v_o(t) = \underbrace{K \cdot V_s}_{V_0} \cos(\omega t + \phi)$

Phase shift

Amp scale



$$v_0(t) = v_s(t) - i(t)R$$

$$i(t) = C\frac{dv_0}{dt}$$

$$v_0(t) = v_s(t) - RC\frac{dv_0}{dt}$$

$$v_s(t) = v_0(t) + \tau\frac{dv_0}{dt}$$

Exponential Representation

- **Euler's Theorem**

$$e^{jwt} = cos(wt) + j\,sin(wt)$$

- $sin(wt)$ **and** $cos(wt)$ **can be represented by linear combination of complex exponential:**

$$cos(wt) = \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right)$$

$$sin(wt) = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$$

Magic: Turn Diff Eq into Algebraic Eq

- **Integration and differentiation are trivial with complex numbers:**

$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \qquad \int e^{i\omega\tau}d\tau = \frac{1}{i\omega}e^{i\omega t}$$

- **Any ODE is now trivial algebraic manipulations … in fact, we'll show that you don't even need to directly derive the ODE by using phasors**

- **The key is to observe that the current/voltage relation for any element can be derived for complex exponential excitation**

Solving LPF with Phasors

- **Let's excite the system with a complex exp:**

$$v_s(t) = v_0(t) + \tau \frac{dv_0}{dt}$$

use *j* to avoid confusion

$$v_s(t) = V_s e^{j\omega t}$$

$$v_o(t) = |V_0| e^{j(\omega t + \phi)} = V_0 e^{j\omega t}$$

real **complex**

$$V_s \cancel{e^{j\omega t}} = V_0 \cancel{e^{j\omega t}} + \tau \cdot j\omega \cdot V_0 \cancel{e^{j\omega t}}$$

$$V_s = V_0 (1 + j\omega \cdot \tau)$$

$$\frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

Easy!!!

Magnitude and Phase Response

- The system is characterized by the complex function

$$H(\omega) = \frac{V_0}{V_s} = \frac{1}{(1 + j\omega \cdot \tau)}$$

- The magnitude and phase response match our previous calculation:

$$|H(\omega)| = \left|\frac{V_0}{V_s}\right| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\measuredangle H(\omega) = -\tan^{-1} \omega\tau$$

Why did it work?

- **Again, the system is linear:**

$$y = \mathbf{L}(x_1 + x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

- **To find the response to a sinusoid, we can find the response to $e^{i\omega t}$ and $e^{-i\omega t}$ and sum the results:**

$$e^{i\omega t} \xrightarrow{\text{LTI System H}} |H(\omega)|e^{i(\omega t+\phi_1)}$$

$$e^{-i\omega t} \xrightarrow{\text{LTI System H}} |H(-\omega)|e^{i(-\omega t+\phi_2)}$$

$$\frac{e^{i\omega t}+e^{-i\omega t}}{2} \xrightarrow{\text{LTI System H}} \frac{H(\omega)e^{i\omega t}+H(-\omega)e^{-i\omega t}}{2}$$

(cont.)

- **Since the input is real, the output has to be real:**

$$y(t) = \frac{H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}}{2}$$

- **That means the second term is the conjugate of the first:**

$$\left|H(-\omega)\right| = \left|H(\omega)\right| \qquad \text{(even function)}$$

$$\prec H(-\omega) = -\prec H(\omega) = -\phi \qquad \text{(odd function)}$$

- **Therefore the output is:**

$$y(t) = \frac{\left|H(\omega)\right|}{2}\left(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right)$$

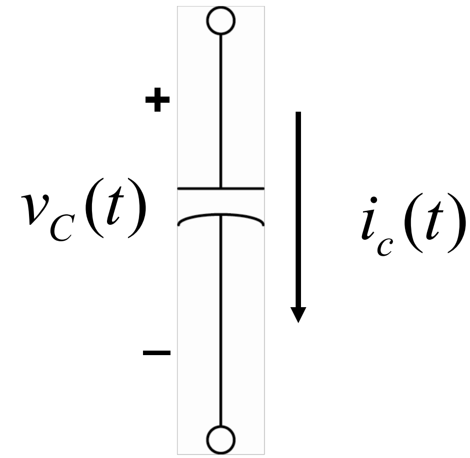
$$= \left|H(\omega)\right|\cos(\omega t + \phi)$$

Phasors

- **With our new confidence in complex numbers, we go full steam ahead and work directly with them … we can even drop the time factor $e^{i\omega t}$ since it will cancel out of the equations.**
- **Excite system with a phasor:** $\tilde{V}_1 = V_1 e^{j\phi_1}$
- **Response will also be phasor:** $\tilde{V}_2 = V_2 e^{j\phi_2}$
- **For those with a Linear System background, we're going to work in the frequency domain**
  - **This is the Laplace domain with** $s = j\omega$

Capacitor I-V Phasor Relation

- **Find the Phasor relation for current and voltage in a cap:**



$$i_c(t) = C\frac{dv_C(t)}{dt}$$

$$i_c(t) = I_c e^{j\omega t}$$

$$v_c(t) = V_c e^{j\omega t}$$

$$I_c e^{j\omega t} = C\frac{d}{dt}[V_c e^{j\omega t}]$$

$$CV_c\frac{d}{dt}e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c e^{j\omega t} = j\omega CV_c e^{j\omega t}$$

$$I_c = j\omega CV_c$$

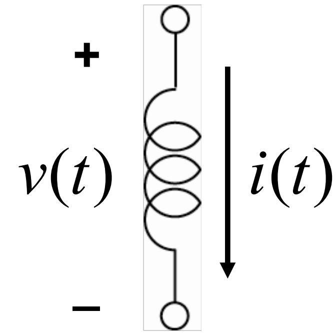
Inductor I-V Phasor Relation

- **Find the Phasor relation for current and voltage in an inductor:**

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = Ie^{j\omega t}$$

$$v(t) = Ve^{j\omega t}$$



$$Ve^{j\omega t} = L\frac{d}{dt}[Ie^{j\omega t}]$$

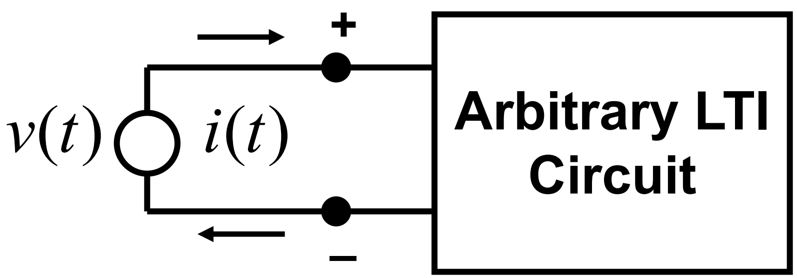
$$LI\frac{d}{dt}e^{j\omega t} = j\omega LIe^{j\omega t}$$

$$Ve^{j\omega t} = j\omega LIe^{j\omega t}$$

$$V = j\omega L\,I$$

Impede the Currents !

- **Suppose that the "input" is defined as the voltage of a terminal pair (port) and the "output" is defined as the current into the port:**



$$v(t) = Ve^{j\omega t} = |V|e^{j(\omega t + \phi_v)}$$

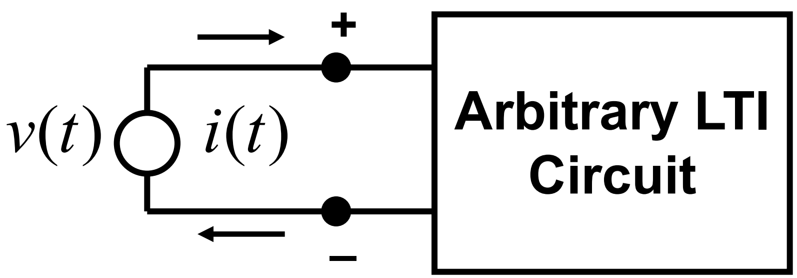
$$i(t) = Ie^{j\omega t} = |I|e^{j(\omega t + \phi_i)}$$

- **The impedance Z is defined as the ratio of the phasor voltage to phasor current ("self" transfer function)**

$$Z(\omega) = H(\omega) = \frac{V}{I} = \left|\frac{V}{I}\right| e^{j(\phi_v - \phi_i)}$$

Admit the Currents!

- Suppose that the "input" is defined as the current of a terminal pair (port) and the "output" is defined as the voltage into the port:



$$v(t) = Ve^{j\omega t} = |V|e^{j(\omega t + \phi_v)}$$

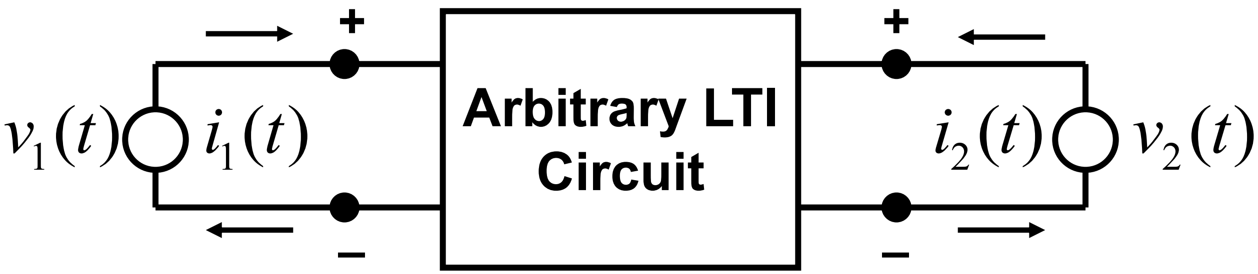
$$i(t) = Ie^{j\omega t} = |I|e^{j(\omega t + \phi_i)}$$

- The admittance Y is defined as the ratio of the phasor current to phasor voltage ("self" transfer function)

$$Y(\omega) = H(\omega) = \frac{I}{V} = \left|\frac{I}{V}\right| e^{j(\phi_i - \phi_v)}$$

Voltage and Current Gain

- The voltage (current) gain is just the voltage (current) transfer function from one port to another port:



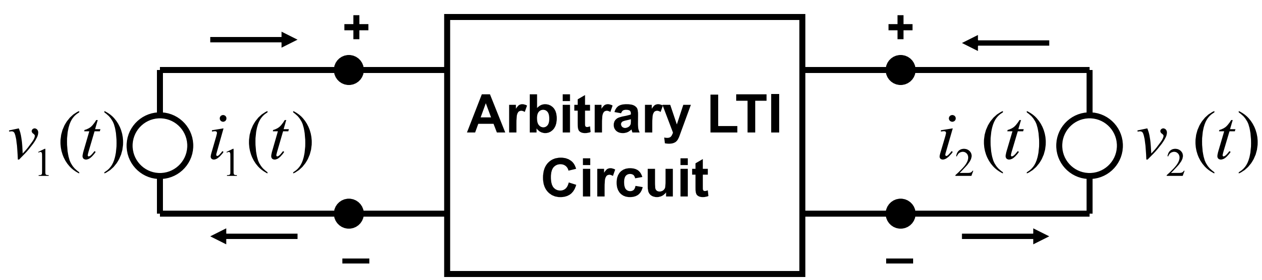
$$G_v(\omega) = \frac{V_2}{V_1} = \left|\frac{V_2}{V_1}\right| e^{j(\phi_2-\phi_1)}$$

$$G_i(\omega) = \frac{I_2}{I_1} = \left|\frac{I_2}{I_1}\right| e^{j(\phi_2-\phi_1)}$$

  – If $|G| > 1$, the circuit has voltage (current) gain
  – If $|G| < 1$, the circuit has loss or attenuation

Transimpedance/admittance

- Current/voltage gain are unit-less quantities
- Sometimes we are interested in the transfer of voltage to current or vice versa



$$J(\omega)=\frac{V_2}{I_1}=\left|\frac{V_2}{I_1}\right|e^{j(\phi_2-\phi_1)} \qquad [\Omega]$$

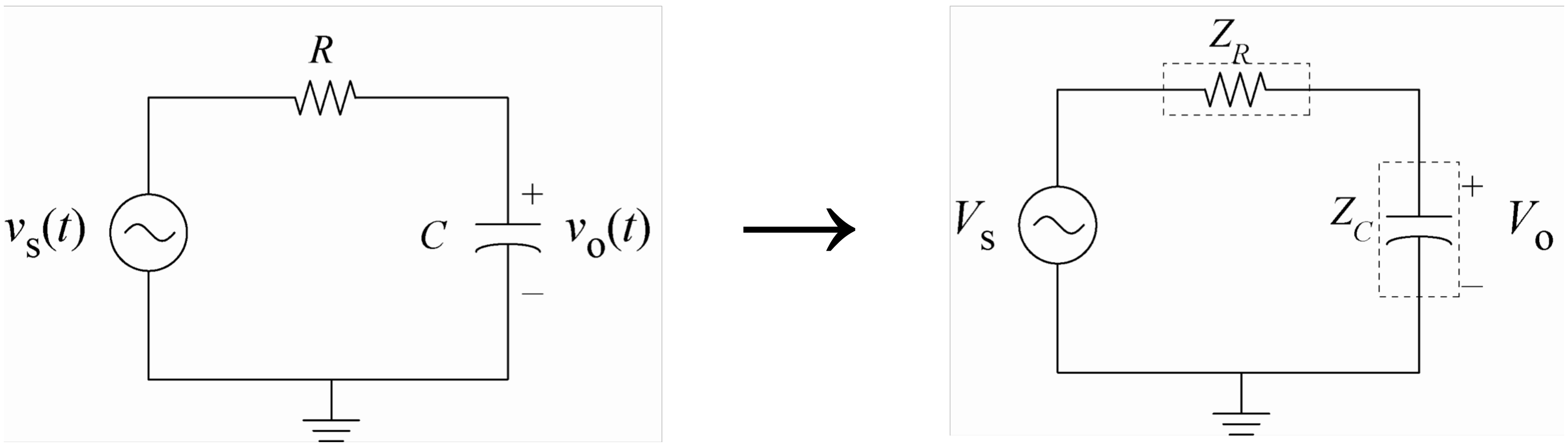
$$K(\omega)=\frac{I_2}{V_1}=\left|\frac{I_2}{V_1}\right|e^{j(\phi_2-\phi_1)} \qquad [S]$$

Direct Calculation of *H* (no DEs)

- **To directly calculate the transfer function (impedance, trans-impedance, etc) we can generalize the circuit analysis concept from the "real" domain to the "phasor" domain**
- **With the concept of impedance (admittance), we can now directly analyze a circuit without explicitly writing down any differential equations**
- **Use KVL, KCL, mesh analysis, loop analysis, or node analysis where inductors and capacitors are treated as complex resistors**

LPF Example: Again!

- **Instead of setting up the DE in the time-domain, let’s do it directly in the frequency domain**
- **Treat the capacitor as an impedance:**



time domain “real” circuit

frequency domain “phasor” circuit

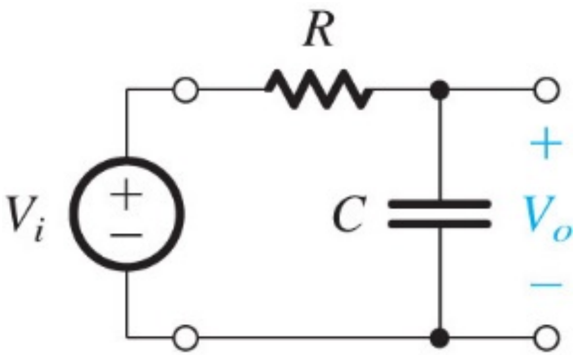
- **We know the impedances:**

$$Z_R = R \qquad\qquad Z_C = \frac{1}{j\omega C}$$

Bode Plots

- **Simply the log-log plot of the magnitude and phase response of a circuit (impedance, transimpedance, gain, …)**
- **Gives insight into the behavior of a circuit as a function of frequency**
- **The "log" expands the scale so that breakpoints in the transfer function are clearly delineated**

Frequency Response of Low-Pass Filters



$$T(\omega)=\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}=\frac{1}{1+j\omega RC}=\frac{1}{1+j\omega/\omega_0}$$

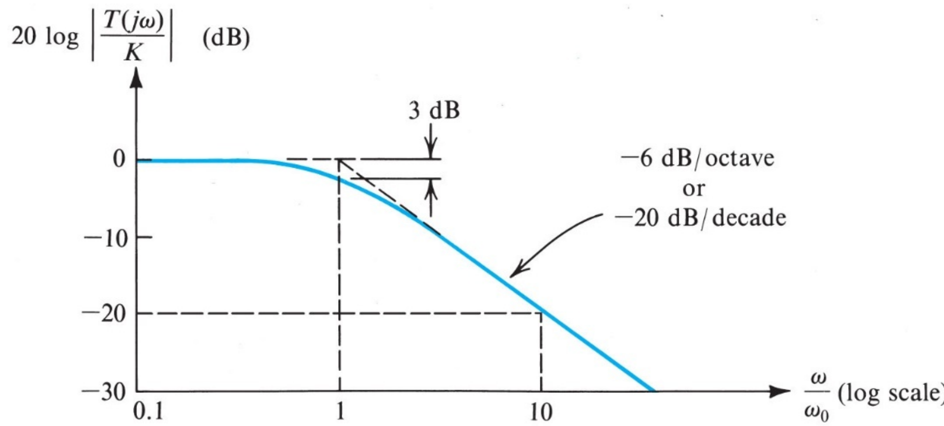
$$\omega_0=\frac{1}{RC}$$

$$\left|T(\omega)\right|=\frac{1}{\sqrt{1+\left(\omega/\omega_0\right)^2}}$$

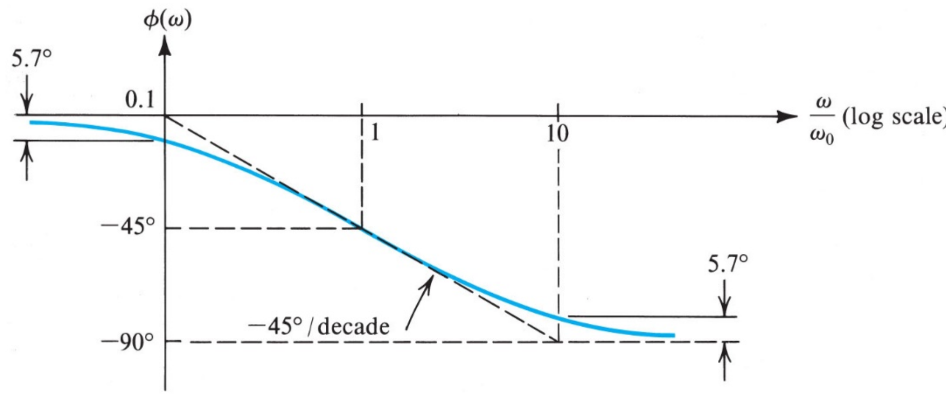
$$\angle T(\omega)=-\tan^{-1}\left(\omega/\omega_0\right)$$

$$\omega_{3dB}=\omega_0 \qquad \text{[rad/sec]}$$

$$f_{3dB}=\frac{\omega_0}{2\pi} \qquad \text{[Hz]}$$

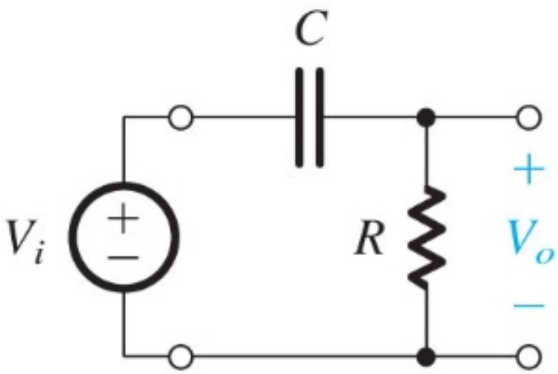


(a)



(b)

Frequency Response of High-Pass Filters



$$T(\omega)=\frac{R}{R+\frac{1}{j\omega C}}=\frac{1}{1+\frac{1}{j\omega RC}}=\frac{1}{1-j\omega_0/\omega}$$

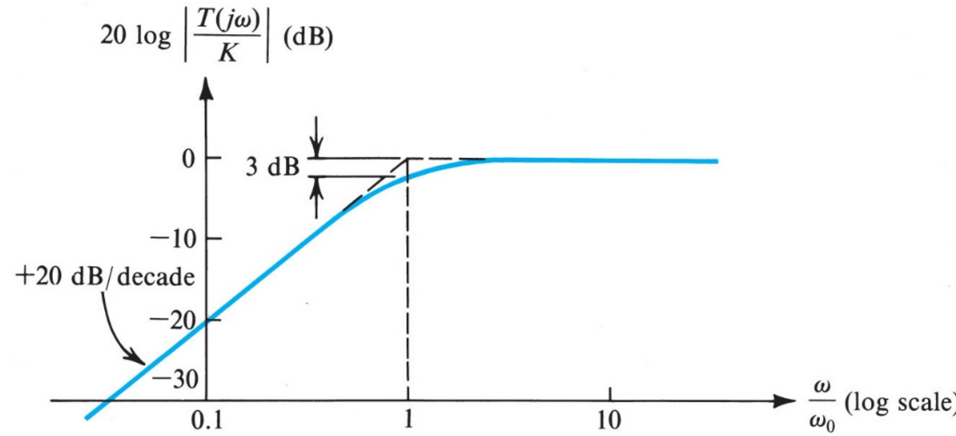
$$\omega_0=\frac{1}{RC}$$

$$|T(\omega)|=\frac{1}{\sqrt{1+(\omega_0/\omega)^2}}$$

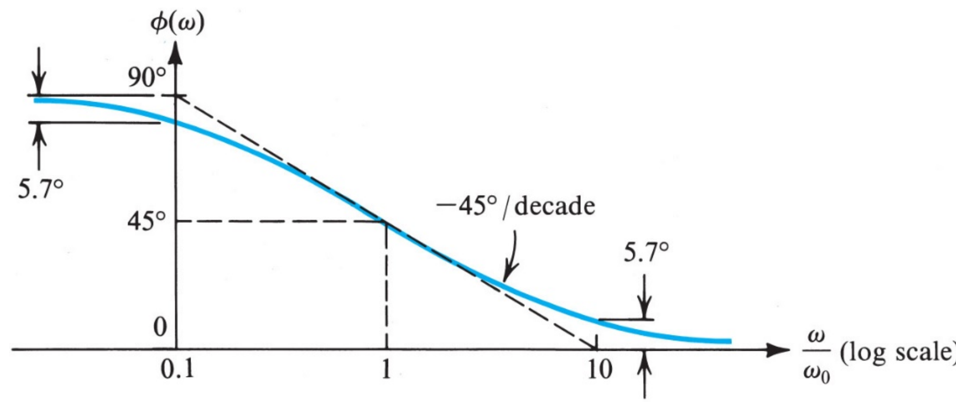
$$\angle T(\omega)=\tan^{-1}(\omega_0/\omega)$$

$$\omega_{3dB}=\omega_0 \qquad \text{[rad/sec]}$$

$$f_{3dB}=\frac{\omega_0}{2\pi} \qquad \text{[Hz]}$$



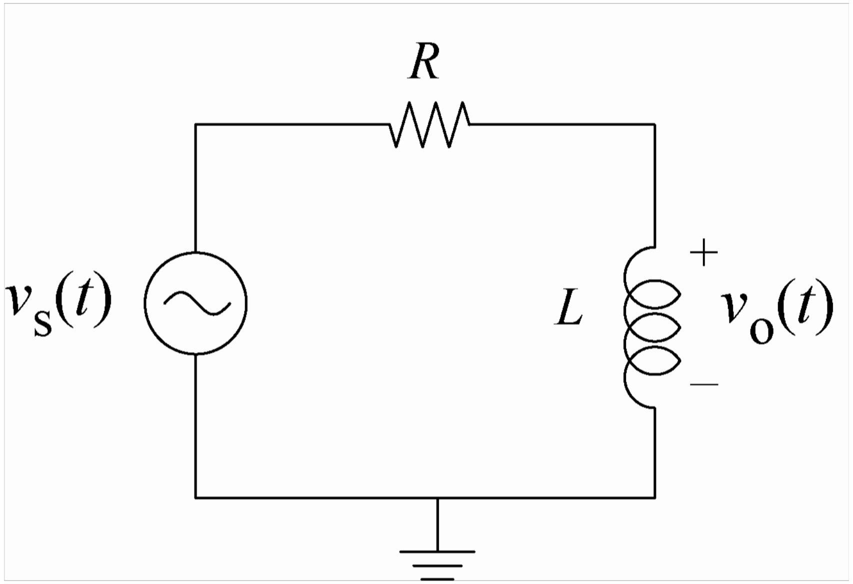
(a)



(b)

Example: High-Pass Filter

- **Using the voltage divider rule:**



$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}}$$

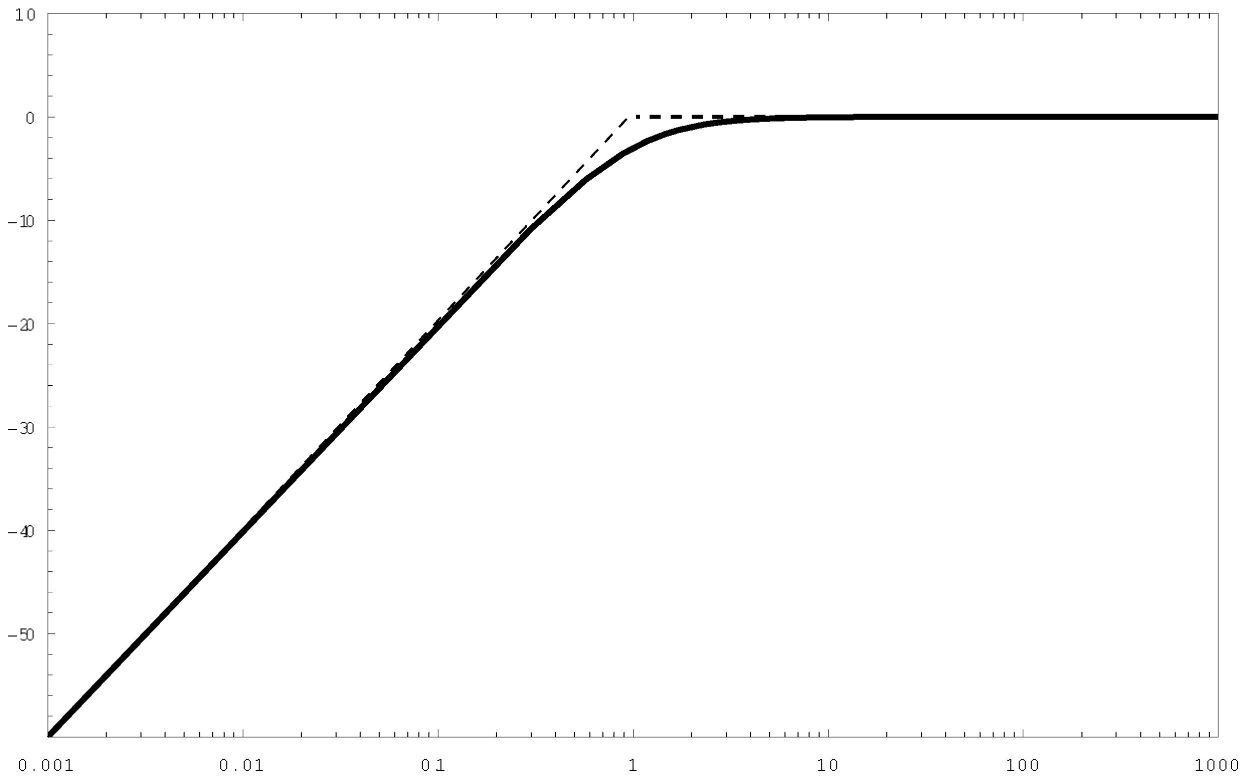
$$H(\omega) = \frac{j\omega\tau}{1 + j\omega\tau}$$

$$\omega \to \infty \qquad |H| \to \left|\frac{j\omega\tau}{j\omega\tau}\right| = 1$$

$$\omega \to 0 \quad |H| \to \frac{0}{1+0} = 0$$

$$\omega = \frac{1}{\tau} \quad |H| = \left|\frac{j}{1+j}\right| = \frac{1}{\sqrt{2}}$$

Approximate versus Actual Plot



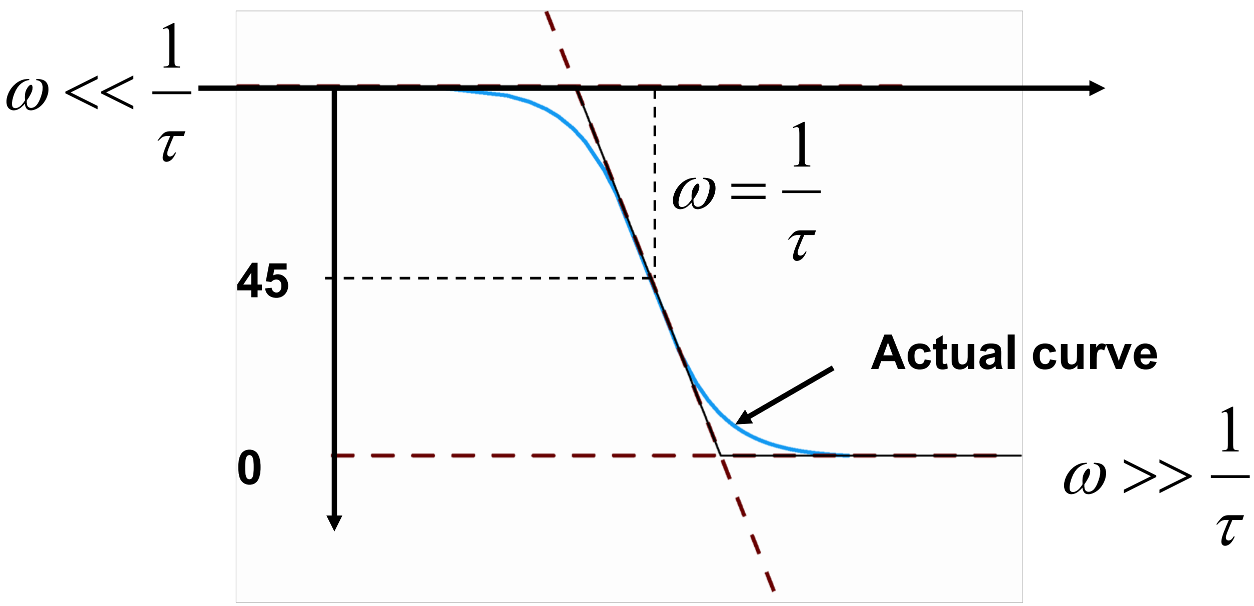
- **Approximate curve accurate away from breakpoint**
- **At breakpoint there is a 3 dB error**

HPF Phase Plot

- **Phase can be naturally decomposed as well:**

$$\angle H(\omega) = \angle \frac{j\omega\tau}{1+j\omega\tau} = \angle j\omega\tau + \angle \frac{1}{1+j\omega\tau} = \frac{\pi}{2} - \tan^{-1}\omega\tau$$

- **First term is simply a constant phase of 90 degrees**
- **The second term is the arctan function**
- **Estimate arctan function:**



Power Flow

- **The instantaneous power flow into any element is the product of the voltage and current:** $P(t) = i(t)v(t)$
- **For a periodic excitation, the average power is:**

$$P_{av} = \int_T i(\tau)v(\tau)d\tau$$

- **In terms of sinusoids we have**

$$P_{av} = \int_T |I|\cos(\omega t + \varphi_i)|V|\cos(\omega t + \varphi_v)d\tau$$

$$= |I|\cdot|V|\int_T (\cos\omega t\cos\varphi_i - \sin\omega t\sin\varphi_i)\cdot(\cos\omega t\cos\varphi_v - \sin\omega t\sin\varphi_v)d\tau$$

$$= |I|\cdot|V|\int_T d\tau\cos^2\omega t\cos\varphi_i\cos\varphi_v + \sin^2\omega t\sin\varphi_i\sin\varphi_v + \cancel{c\sin\omega t\cos\omega t}$$

$$= \frac{|I|\cdot|V|}{2}(\cos\varphi_i\cos\varphi_v + \sin\varphi_i\sin\varphi_v) = \frac{|I|\cdot|V|}{2}\cos(\varphi_i - \varphi_v)$$

Power Flow with Phasors

$$P_{av} = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v)$$

Power Factor

- **Note that if** $(\phi_i - \phi_v) = \frac{\pi}{2}$ **, then** $P_{av} = \frac{|I| \cdot V}{2} cos\left(\frac{\pi}{2}\right) = 0$
- **Important: Power is a non-linear function so we can't simply take the real part of the product of the phasors:**

$$P \neq \mathrm{Re}[I \cdot V]$$

- **From our previous calculation:**

$$P = \frac{|I| \cdot |V|}{2} \cos(\phi_i - \phi_v) = \frac{1}{2} \mathrm{Re}[I \cdot V^*] = \frac{1}{2} \mathrm{Re}[I^* \cdot V]$$

Summary

- Complex exponentials are eigen-functions of LTI systems
  - Steady-state response of LCR circuits are LTI systems
  - Phasor analysis allows us to treat all LCR circuits as simple "resistive" circuits by using the concept of impedance (admittance)
- Frequency response allows us to completely characterize a system
  - Any input can be decomposed into either a continuum or discrete sum of frequency components
  - The transfer function is usually plotted in the log-log domain (Bode plot) – magnitude and phase
  - Location of poles/zeros is key